Name of College - S.S. College J. Bad
Deptt - Mathematics
Topic - Problem based on Euler's Theorem (Partial Differentiation) (Contd)
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Total Differential :→

Let $u = f(x, y)$ be a function of two variables x and y.
Let us suppose that u takes up the value $u + \Delta u$ when x takes up the value $x + \Delta x$ and y, $y + \Delta y$ so that $\Delta x, \Delta y \to 0$, $\Delta u \to 0$.

$$\therefore \quad u + \Delta u = f(x+\Delta x, y+\Delta y)$$

$$\therefore \quad \Delta u = f(x+\Delta x, y+\Delta y) - f(x, y)$$

$$= f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x, y)$$

$$= \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x} \cdot \Delta x + \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} \Delta y$$

By Mean value theorem, we have

$$\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x} = \frac{\partial}{\partial x} f(x+\theta_1 \Delta x, y+\Delta y)$$
where $0 < \theta_1 < 1$

and
$$\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \frac{\partial}{\partial y} f(x, y+\theta_2 y) \quad \text{—— (11)}$$
where $0 < \theta_2 < 1$

$$\therefore \quad \delta u = \frac{\partial}{\partial x} f(x+\theta_1 \Delta x, y+\Delta y)\, \Delta x + \frac{\partial}{\partial y} f(x, y+\theta_2 \Delta y)\, \Delta y$$

Now we consider $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous

in the nbd of point (x,y),
therefore we may have

$$\frac{\partial}{\partial x} f(x+\theta_1 \Delta x, y+\Delta y) = \frac{\partial}{\partial x} f(x,y) + \epsilon_1$$

and $$\frac{\partial}{\partial y} f(x, y+\theta_2 \Delta y) = \frac{\partial}{\partial y} f(x,y) + \epsilon_2$$

where $\epsilon_1 \to 0$ $\epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

hence from (II)

$$\Delta u = \frac{\partial}{\partial x} f(x,y)\Delta x + \frac{\partial}{\partial y} f(x,y)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \quad \text{(III)}$$

therefore from (III) in the limiting stage when
$\Delta x \to 0$ $\Delta y \to 0$, $\Delta u \to 0$

$$du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Exact Differential

If an Expression of the Type
$M\,dx + N\,dy$ where
M and N are functions
of x and y or constant may be reduced to du,
where u is the function of x and y, then
$M\,dx + N\,dy$ is said to be Exact

Show that the necessary and sufficient condition that the Expression $M\,dx + N\,dy$ be an Exact differential is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$M\,dx + N\,dy$ — (1)

The condition is necessary:→ Let the Given Expression be an Exact differential and let it be equal to du

Now $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

$\Rightarrow M\,dx + N\,dy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

Equating the co-efficients of dx and dy separately from both sides, we get -

$$M = \frac{\partial u}{\partial x} \qquad N = \frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$, Assuming that, under certain conditions

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{--- (II)}$$

Condition is sufficient :-

We are to show if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

then $M dx + N dy$ will be an Exact differential

Let y = constant $\Rightarrow dy = 0$

LHS becomes $M dx$. Let $\int M dx = v$.

$$\Rightarrow \frac{\partial v}{\partial x} = M \quad \text{--- (III)}$$

From (II) $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial^2 v}{\partial x \partial y}$

or $\frac{\partial N}{\partial x} - \frac{\partial^2 v}{\partial x \partial y} = 0$

$$\Rightarrow \frac{\partial}{\partial x}\left(N - \frac{\partial v}{\partial y}\right) = 0$$

This shows that the function $N - \frac{\partial v}{\partial y}$ is constant so far as x is concerned and is therefore a function of y only.

Denoting the value by $f'(y)$, we have

$$N - \frac{\partial v}{\partial y} = f'(y)$$

$$\Rightarrow N = \frac{\partial v}{\partial y} + f'(y) \qquad \text{--- (IV)}$$

Hence if we write

$$u = v + f(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} \text{ and } \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}$$

then by (iii) and (iv) we get -

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = M \quad \text{—— (vii)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + f'(y) = N \quad \text{—— (viii)}$$

$$\therefore M\,dx + N\,dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow M\,dx + N\,dy = du$$

Hence the condition is sufficient.